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TWO STREAM POLLUTION CONTROL MODELS

by

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I. INTRODUCTION

Since Pigou's [13] analysis, economists have viewed pollution as a particular problem of externalities, the cure of which was dependent on finding optimal taxes and/or subsidies or defining property rights and setting up appropriate markets for the exchange of these rights. The former prescription is attributable to Pigou while the latter is generally attributable to Coase [6]. Other prescriptions have developed as modifications or clarifications of these two remedies.[1/]

It has become evident that such remedies for cases of market failure may be inadequate to achieve a desirable quality of the environment. Economists have been convinced that such remedies can effect the desired results but policy makers are much less trusting of such schemes as they are of standards and other forms of control. Policy makers become concerned about the behavior of society and individuals under such incentive systems and would rather opt for a command behavior via legislation rather than rely on incentives to induce pollution control behavior. This author does not harbor that same distrust, but is willing to admit that an approach beyond the

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1/ c. f. Kneese and Bower [11], Buchanin and Stubblebine [5], Dolbear [8], Crocker and Rogers [7], Tybout [15] and Randall [14], with the exception of Boyd and Mohring [3] who adopt Knight's [12] concept of asset utilization to the externalities problem.

usual market failure analysis may be in order where matters of pollution are involved.

In the process of controlling stream pollution, certain types of productive processes may have to be prohibited, some consumption curtailed, and perhaps some restrictions placed on output and population growth. It is not the purpose of this paper to develop a theory of optimal growth which incorporates output and population growth restrictions, however, the intertemporal aspects of limiting pollution production and production process choice will be the central thrust of the paper. We will view the pollution problem as a pervasive phenomenon where physical substances impair the enjoyment of life by individuals, or by society in general. In particular we develop two models of optimal stream pollution control with due consideration of the intertemporal aspects of the problem. The two models of control are differentiated depending on whether control through pollution reduction or choice of production is considered. The developments will be in the tradition of the dynamics of Arrow [1].

II. CONTROL THROUGH POLLUTION REDUCTION

The capacity to enjoy life is assumed to be limited or enhanced by the stocks of physical substances; i.e., the pollution problem is a stock problem. The stock enters the welfare function with a negative marginal utility but has a positive marginal product in production. The stock is detrimental to consumption directly or because production is impaired. We are unable to control the stock directly, thus, in the tradition of instruments and state variables in dynamic control theory, no relationship can be assumed between state variables (pollutant free environments, output, etc.) and the stock. The control path becomes a planning path, i.e., planning is imperative to direct the growth path of the undesirable substance.

In this context, no planning (ignoring the problem hoping it will fade away as a flow problem) may result in a stock path which becomes prohibitive to any enjoyment of life, i.e., zero consumption or complete impairment of production. Overreaction to the problem results in large quantities of resources being employed in pollution reduction which otherwise could be employed in other uses raising the welfare to society more than the reduction of pollution depending on the approach path of the stock.

We assume an objective function defined for the utility of society expressed as a function of consumption and pollution (reduction of consumption,)

(1) $$W = W(C,Z)$$

Consumption, C, is a flow while pollution, Z, enters the welfare function as a stock. We assume W has continuous first and second derivatives, $W_c > 0$, $W_{cc} < 0$, $W_z < 0$, $W_{zz} < 0$, and further that $W_c = \infty$ for $C = 0$. 2/ Originally we do not specify the appropriate structure of the objective function.

If we apply a discount rate, r, to the flow of utility, the total welfare associated with any time path for C and Z is derived by integrating over the time horizon indicated by,

(2) $$\underset{\text{Max}}{W_T} = W\int_0^{\infty}(C,Z)\ e^{-rt}dt.$$

Now let us assume there are two factors of production, labor and a single aggregate capital good. Output is allocated to consumption, capital accumulation, and to the reduction of the stock of Z. The production function expressing output as a function of capital stock is,

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2/ $W_i : \partial W/\partial i$, and $W_{ii} = \partial^2 W/\partial i$ where $i = c,z$. Indifference curves are guaranteed to have the desirable curvature if $W_{cc}W_{zz}-W_{cz}^2 > 0$. Non-inferiority, which is assumed, requires $W_{cc}-W_{cz}W_c/W_z < 0$ and $W_{zz}W_c/W_z-W_{cz} > 0$.

(3) $Q = f(K)$,

since we assume a fixed labor supply. Output equals Q, and the capital stock is K, and depreciates at a rate $d > 0$.

The pollutant in this model serves no productive purpose. The flow of the pollutant is assumed to be a fixed proportion by-product of production (joint production). 3/ The stock of pollution, Z, deteriorates naturally at some rate $g \geq 0$. 4/

Pollution may also be reduced through control expenditures as was indicated above. To analyze this method of pollution reduction let a and b be non-negative fractions of output allocated to consumption and pollution reduction respectively. 5/ The growth rate of the capital stock is expressed as,

(4) $\dot{K} = (1-a-b) \cdot f(K) - d \cdot K$

The rate of growth of the stock of the pollutant can be epxressed as,

(5) $\dot{Z} = (1-bv) \cdot f(K) - bZ$

Where we assume constant returns in the pollution reduction efforts; i.e., one unit of output will reduce the pollutant by v units. 6/

The model is essentially a two control variable model since output is allocated to only consumption, capital accumulation and pollution control.

3/ A measure of pollution in the same units as output can be adopted since the joint production relationship is assumed.

4/ The stock has a depreciation rate, but the more applicable physical-biological term is natural deterioration as defined for our purposes.

5/ Then, $C = af(K)$, under this specification.

6/ The capital stock and pollution stock are non-negative.

We center control on consumption and pollution control, and capital accumulation will be determined once the optimal paths for the first two are determined. The necessary conditions for an optimal path are obtained using the method of optimal control and similar notation as Bryson and Ho [4].

Let k and z be the shadow prices at any point in time for capital and pollution respectively. The Hamiltonian, H, becomes,

$$H = e^{-rt}\left\{ [\, W(C,Z) + k(1-a-b)\, f - d\cdot K] + z\, [(1-bv)\, f - bz] \right\} \tag{6}$$

By applying Pontryagin's maximum principle a maximum will be achieved if the Hamiltonian is maximized. That is the objective- to maximize the total discounted welfare. The maximum is achieved provided the shadow prices are defined by the conditions,

$$\dot{k} = -H_k + k_r = -W_c\, af' + K[r+d - (1-a-b)F'] - Z(1-bv)F' \tag{7}$$

$$Z = -H_z + zr = -W_z + (v+g)z \tag{8}$$

and initial prices k(0), z (0) are properly chosen. Our problem is one of choice of a and b for which H is a maximum. Taking partial derivatives,

$$H_a = e^{-rt}(W_c - k)f \tag{9}$$

$$H_b = e^{-rt}(-vz - k)f \tag{10}$$

a and b must be chosen such that $H_a = \text{Max}\,(H_b, 0)$ along the optimal path since H is linear in a,b. 7/

There are at most two steady states derived from the model; i.e., optimal paths when K and Z are held constant. The steady states derived are interesting in a policy context. At either equilibrium $\dot{K}$ and $\dot{Z}$ must

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7/ All output could conceptually be expended solely on consumption and pollution control. A constraint on such a solution would modify the model for $a+b \leq A$. A non-negative time dependent parameter, $\gamma(t)$, exists in the constrained case, such that $H_a = H_b + \gamma(t)$, with $\gamma(t)\cdot(a-A) = 0$ defined for all t.

be zero and this implies,

(11) $\quad f(K)\ (1-a-b) = d\cdot K$

(12) $\quad f(K)\ (1-bv) = g\cdot Z$

Now, if we multiply (11) by v and subtract from (12), we get,

(13) $\quad va\ f(K) = vC = g\cdot Z - dgK + (v-1)\ f(K).$

Since K is fixed, and $d > 0$, $a+b < 1$ and is constant. Hence, $H_a = 0$, which implies $H_b \leq 0$, and

(14) $\quad W_c\ (a\ f(K),\ Z) = k$

which in turn implies that the shadow prices, k, z, are constant, upon substitution of (14) into the perfect foresight condition (7), we get the following expression for $\dot{k}$.

(15) $\quad \dot{k} = k\ (r+d-f') - zf' + b(zv-k)f'$

Two interesting cases are derived (steady states).

<u>Case 1</u> (No expenditure for pollution reduction):

If $H_b < 0$, $b = 0$), and we can solve (15) to obtain,

(16) $$f'\ (K) = \frac{r+d}{1+z/k} = \frac{r+d}{1+\frac{W_z}{W_c\ (r+g)}}$$

where Z is implicitly defined as a function of K, Z (K). We can solve for Z' (K),

(17) $$Z'(K) = \frac{C_1 f'' - C_2\ af'\ [W_{cc}W_z - W_c W_{cz}]}{W_{cz}W_z - W_c W_{zz}}$$

where C_1 and C_2 are positive factors. Since (12) is upward sloping and by assumption (noninferiority) $Z' < 0$, there is at most one solution.

<u>Case 2</u> (Some expenditure on Pollution Control and some on Consumption):

If $H_b = 0$, $z = -k/v$ and,

(18) $\dot{k} = k(r+d-(1-1/v)f') = 0,$

which is satisfied when

(19) $f'(K) = r+a/1-\frac{1}{v}$

The K satisfying (19) is unique since $f'' < 0$.[8] However, $\dot{z} = 0$, so

(20) $W_z = (r+g)z = -W_c \frac{(r+g)}{v}$

Again by assumption the curve in C,Z space which satisfies (20) is downward sloping and (13) is upward sloping and at most one solution occurs.

This solution derives consumption-pollution paths which are analogous to the "Golden Age" paths in growth dynamics as illustrated in Figure 1.

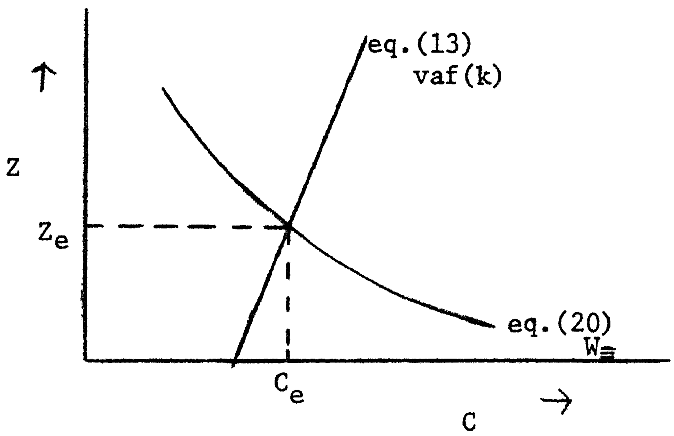

Figure 1. Equilibrium for Case 2

The values of K and Z which trace out the optimal path (pollution-consumption) when they are held constant are the solutions to equations

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[8] Assuming such a K exists.

(13) and (20) and, as illustrated, define an equilibrium at the points Z_e and C_e in the pollution-consumption plane.

There can be no more than two K,Z equilibrium pairs that satisfy the necessary conditions for optimality. The equilibrium for Case 1 (no expenditure for pollution control) exists if the value of K satisfying (12) and (17) is greater than the K defined for the equilibrium of Case 2, i.e., in equation (19). The equilibrium of Case 2 exists if (19) is satisfied and if the C obtained by solving (13) and (20) lies in the open interval $0 < c < f(K) - d \cdot K$. If capital is fixed at K, the optimal path is traced from the value Z(0) to the Z resulting from the equilibrium of Case 2, i.e. for any fixed K the optimal path is traced directly to the level of Z for which $W_z = -W_c \cdot [(r+g)v]$.

III. CONTROL BY CHOICE OF PRODUCTION PROCESS

The second model to be derived assumes that pollution enters the welfare function as a stock with a negative marginal utility. The flow is assumed to enter the production function with a positive marginal product. We also assume that the pollution produced deteriorates naturally. Labor is assumed to be the only scarce factor of production and, hence, labor must be allocated to employment in the consumption and pollution sectors. Let the respective labor allocation be N_c and N_z.

The objective function is the same as for the first model developed above but a form for the utility function is assumed which is separable in the arguments, consumption and pollution, i.e.,

(21) $\qquad W(C,Z) = h(C) - \ell(z),$

for $h' \geq 0$, $h'' < 0$, $h'(0) = +\infty$, $\ell' \geq 0$, $\ell'' > 0$, $\ell'(0) = 0$. The production function for consumption goods takes the form,

(22) $C = (N_1, q(N_2))$,

where $q(N_2)$ defines the flow of the produced pollutant as a function of the labor allocated to the pollution sector. Pollution serves as an intermediate good in the production function expressed in (22). It is assumed that $q'(N_2) > 0$, $q''(N_2) < 0$ for $0 \leq N_2 \leq N$, where N is the total fixed labor supply.

The production function (22) can be reduced to,

(23) $C = f(N_2)$,

since a positive amount of labor input is needed for any production of the consumption good. The total amount of labor, N, then is a given parameter at some level. The production function (23) as a function of N_2, the labor allocated to pollution production (production which also results in creation of pollution), increases, but beyond some maximum level of N_2, N'_2, increased labor allocation to pollution production reduces the output of consumption goods. The loss from the reduced direct labor input is greater than the gain from the increased pollution input.

Let us now look at the necessary conditions. The modified Hamiltonian can be expressed as,

(24) $H = e^{-rt} \; [h \; (f(N_2) - \ell(Z) + z(d(N_2) - gz) + p \, N_1 + sN_2$,

where p and s are time dependent non-negative multipliers corresponding to the consumption and pollution labor input constraints. We know that $N_2 < N$ and $p = 0$ since no more than N'_2 of the labor input will be employed in pollution production. 9/ The multiplier, s, may be positive however.

9/ Pollution as created by production processes is valued nonpositively.

Given the pollution shadow price z, the optimal value for N_2 can be derived from,

(25) $$H_{N_2} = h'f' + zq' + s = 0$$

satisfying $s \geq 0$ and $sN_2 = 0$. If $z = 0$, then $N_2 = N'_2 > 0$ and hence $s = 0$. If we differentiate (25) with respect to z we obtain,

(26) $$dN_2/dz = -q' \; h''f + h'f'' + q''H)$$

This is greater than zero if $q'' < h'' \; , f''$. Let us assume this to be the case. 10/

Since $f(N_2) > 0$ at $N_2 = 0$, then $h'\ (f(0))$ is finite. For $s = 0$ there exists a lower bound on the pollution shadow price given by,

(27) $$\underline{z} = -[h'\ (f(0))\ f'(0)]/q'(0).$$

Equation (25) defines s for $z < \underline{z}$ and becomes positive. However, in this region $N_2 = 0$.

Now the steady state conditions can be derived given the perfect foresight condition for z as,

(28) $$\dot{z} = (p+g)z + \ell'$$

Adopting the phase diagram approach Figure 2 illustrates the dynamic conditions.

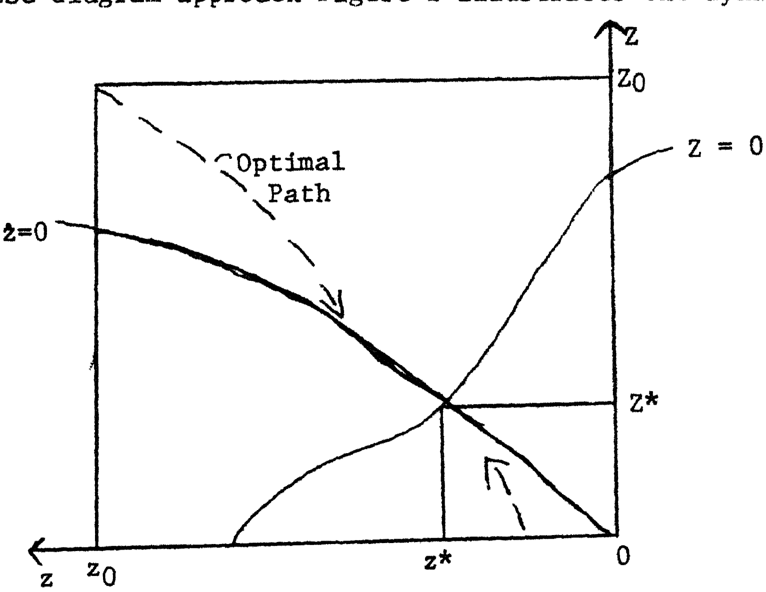

Figure 2. Dynamic Paths of Conditions (28) and (29) and the Optimal Path

10/ This is a reasonable assumption for pollutants where resources spent in their production are small relative to total production.

The path $z = 0$ passes through the origin with slope $-(p+g) \ / \ 1 \ (z) \ \ 0$. The path $Z = 0$ is defined by,

(29) $$Z = q \ (N,z)/g$$

Since $g>0$, $q>0$, $Z = 0$ has positive slope and the two paths have a unique intersection, (z^*, Z^*) and the optimal path exists and runs through the intersection [11/] the optimal initial level of the pollution shadow price, z_o, is the value on the optimal path corresponding to some initial level, Z_o, of pollution.

IV. APPLICATIONS OF THE TWO MODELS TO STREAM POLLUTION CONTROL

The model of pollution control by reduction of pollution by abatement expenditure (the model developed in section II above) gives different results from the control by choice of production model (section III above). The differences suggest that no general rules exist for pollution control. However, the models do extend to a variety of pollution control problems.

If we consider the problem of stream pollution and policy to effect its control, both models apply but the methods of control are different. The level of pollution (or the reverse, level of environmental quality) is a public good and our objective function used in both models is applicable. A control policy which forces industrial or consumption goods producers to treat their waste water before discharge into the common property resource (stream) forces the adoption of a more expensive production process. The model developed in section II applies.

The paper industry is a good example. It has been estimated that this industry uses approximately 1,300 billion gallons of water for processing per

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11/ We take the proof of the existence of the optimal path as given from Arrow [1].

year and discharges 3 billion pounds of suspended solids. The five-day BOD concentration of the process water is approximately 5.9 billion pounds (second only to the chemical manufacture industries).[12/] The paper industry does have a variety of production processes which emit varying degrees of organic residue in the process water. The less residue that is discharged from a particular process, the less effluent it is in terms of cost per unit of paper output. The results of the choice of process model apply in this case.

The chemical product manufacturing industry which is one of the larger industrial water users, is sometimes limited to one known process. The chemical and chemical-physical interaction properties are limited in the manufacture of some chemical products. However, some chemical industries do have a variety of processes by which the product is produced. Here, both the tools of the pollution-reduction expenditure model and process choice model apply.

If pollutants are discharged into streams and the streams are treated to aid the natural assimilative properties of streams, then the tools of the pollution reduction by expenditure model apply. This is an interesting case in light of the recent controls on matching grants for waste treatment facilities. Those communities obtaining matching grants from the EPA funds must force those industries whose wastes are treated by such a facility to pay part of the costs of the treatment plant, or force them to treat the wastes before discharge. The former case does not force the industries to choose between processes, i.e., choose cheaper processes in terms of the cost per unit of output which emit less pollutants. The latter case forces a pollution flow reduction which may be achieved at a cost in productivity, i.e., through a less productive process if cheaper processes which emit less effluent do not exist.

12/ See the 1968 Cost of Clean Water Report [10].

Earlier we made a stock-flow distinction with regards to pollutants. That distinction is important in light of the results of our two models. Whether a pollutant has its major impact on consumption, production or both and as a stock or flow is important from the standpoint of control policy. Generally pollutants do have a detrimental effect as a stock either because of their direct effect on consumption or because they impair production. The flow generally has a positive marginal product and is therefore beneficial, or the flow is a by-product which can only be removed by abatement expenditures.

The results of our models imply that if the pollutant is an unfortunate by-product flow then the flow is controlled by the use of the pollution reduction method via abatement expenditures. If the pollutant has a positive marginal product in production and is an intermediate good in the production of some final good, then the choice of production process model applies as the control method.

Stream pollutants which are by-products are such substances as whey from dairy product manufacturing, offal from the meat packing and feeds industries, fruit and vegetable canning wastes, metal milling wastes, household sewage and heat as it is transferred to cooling water. Those pollutants which are intermediate goods are such substances as mercury, dies used in paper processing, and pesticides.

Interestingly two control cases derive from the abatement expenditures model. One case allows for pollution control expenditures along with consumption and capital accumulation. The other optimal path is one where an overabundance of capital exists, a high level of consumption is enjoyed and pollution is only controlled by natural forces i.e. a deterioration rate $g > 0$.[13/]

13/ Note that our analysis has not included the pollution by using nonrenewable natural resources at high rates. The question arises as to what is a high rate of use of a nonrenewable resource.

If the deterioration rate is low, then an extreme level of pollution exists under that policy since high levels of consumption and capital accumulation exist, unless, of course, the consumption and production technologies are waste free or no pollutants exist as intermediate goods. The latter would imply a different production function than that which we specified in section II.

We can extend the results of the choice of production process model developed in section III to the interesting control choice of ignoring the pollution problem versus choice of production process which bans the pollutant from use in the production process. Let us return to the phase diagram approach and extend the results of process choice model as illustrated in Figure 3.

Let $\underline{Z}$ be the abscissa of intersection of the optimal path and the line $z = \underline{z}$, where $\underline{z}$ is the lower bound of z derived from the necessary conditions derived in section III. If the initial pollution level, Z_o, is greater than that bound, the optimal policy consists of setting $N_2 = 0$, where N_2 is the labor employed in pollution production. That level of N_2 is maintained until natural deterioration reduces the pollutant level to $\underline{Z}$. Then N_2 is allowed to increase simultaneously maintaining $\dot{Z} < 0$, until the stationary point (z^*, Z^*), which is optimal, is approached. Under this policy, $N_2 < N'_2$, where N'_2 is the level of employment of the labor input in the production of pollution beyond which output declines.

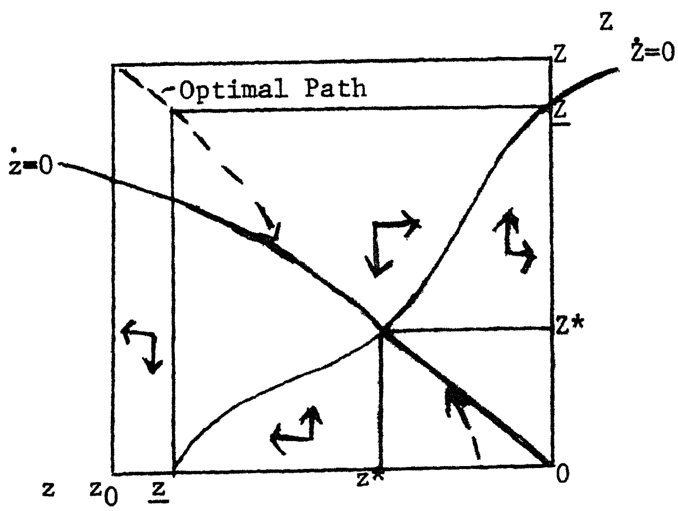

Figure 3. Dynamic Paths for the Optimal and $\dot{z}=0$, $\dot{Z}=0$ Paths

Whether it is an optimal policy to ban a pollutant's use as an intermediate good depends on the level of the stock of the pollutant. The two extreme policies;e.g., to ban the pollutant forever or to ignore the problem, are both non-optimal policies. This is a safe political stance result, but in order to operate the optimal policy, the critical level, $\underline{Z}$, below which the production of the pollutant should be allowed, must be determined. Also, an operational problem of limiting the pollutant to the required amount still remains. That is the reason abolition or permissive policies in their extreme are easier to administrate. However, the fact still remains that the economic losses may be great by choosing such a simple policy when other policies are feasible but the implementation of them has not been tried or tested.

V. FUTURE RESEARCH

The rate of deterioration is crucial in determining the bound beyond which the use of the pollutant should cease. The higher the rate of deterioration the higher is the level at which cessation of the use of the pollutant occurs, and such a policy is likely to be optimal. More research needs to be forthcoming on this issue.

Future research should treat the labor supply as an endogenous variable to assess the relationship between pollution control and population growth, an issue upon which more heat than light has been generated in the recent literature on population control. 14/ In this light, technological change in the pollution removal sector should be analyzed. This would also help to analyze the pollution control problem from the externalities standpoint and assess non-convexities arising from external diseconomies. 15/ Our model should be extended

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14/ c. f. Ehrlich and Ehrlich [9].
15/ c. f. Baumol [2].

to include capital employment in the pollution removal sector. Our models could also be extended to more than production processes and stream polluton per se and the pollutants which enter those processes. With this extension the problems of solid waste build-up could be addressed such as highway litter, packaging, and illegal dumping.

The other interesting problem needing further research is the area of recycling and the use of natural resources. Our models have not included this problem except that our broad definition of the pollution problem does encompass this area of concern. A great deal of work in the dynamics of resource use could be done, particularly in light of the apparent energy crises in which we find ourselves in the U. S.

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